Topology 48 (2009) 197–202

Contents lists available at ScienceDirect

Topology

journal homepage: www.elsevier.com/locate/top

Generalization of the polarization formula for nonhomogeneous polynomials and analytic mappings on Banach spaces

I. Chernega[a,*], A. Zagorodnyuk[b]

[a] Institute for Applied Problems of Mechanics and Mathematics, Ukrainian Academy of Sciences, 3 b, Naukova Street, Lviv 79060, Ukraine

[b] Vasyl Stefanyk Precarpathian National University, 57 Shevchenka Street, Ivano-Frankivsk 76000, Ukraine

ARTICLE INFO

MSC:
46G20
46G25

Keywords:
Multilinear mappings
Polynomials
Analytic mappings on Banach spaces
Generalized Rademacher functions

ABSTRACT

We propose an analogue of the classical polarization formula for any nonhomogeneous polynomial and analytic mapping on complex Banach spaces.

1. Introduction

Let X and Y be linear spaces. A mapping $P : X \to Y$ is called an *n-homogeneous polynomial* if there exists a symmetric n-linear mapping $A : X^n \to Y$ such that $P(x) = A(x, \ldots, x)$. The given symmetric n-linear mapping can be uniquely expressed as a polynomial using the *polarization formula*, which is one of the fundamental results in the theory of polynomials and multilinear mappings. The polarization formula has been known since 1931 [1] but later was rediscovered and published in the works of Martin [2], Mazur, Orlicz [3] and others. The polarization formula has various representations, in particular using the generalized Rademacher functions.

The generalized Rademacher functions were introduced by Aron and Globevnik in [4]. Later, in [5] it was shown that these functions are quite useful in obtaining simple proofs of various estimates in different areas of functional analysis. In this paper we shall use the generalized Rademacher functions to prove an analogue of the polarization formula for nonhomogeneous polynomials on complex linear spaces and for analytic mappings on complex Banach spaces.

For every natural number $n \geq 2$ the generalized Rademacher functions $S_j^{[n]}(t)$ are defined inductively as follows (see [4,5]). Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the complex nth roots of unity. For $j = 1, \ldots, n$ let $I_j = \left(\frac{j-1}{n}, \frac{j}{n}\right)$ and $I_{j_1 j_2}$ denote the j_2th open subinterval of length $\frac{1}{n^2}$ of I_{j_1} $(j_1, j_2 = 1, \ldots, n)$. Proceeding like this, we can define the interval $j_1 j_2 \ldots j_k$ for any k. Now $S_1^{[n]}(t) : [0, 1] \to \mathbb{C}$ is defined by setting $S_1^{[n]}(t) = \alpha_j$ for $t \in I_j$, where $1 \leq j \leq n$. In general, $S_k^{[n]}(t) = \alpha_j$ if t belongs to the subinterval $I_{j_1 j_2 \ldots j_k}$ where $j_k = j$. For all endpoints t we can set $S_k^{[n]}(t) = 1$. Also we set $S_1^{[1]}(t) \equiv 1$.

The basic properties of the generalized Rademacher functions are as follows:

Proposition 1.1 ([5]). 1. *For every $k = 1, 2, \ldots$ and $t \in [0, 1]$, we have $|S_k^{[n]}(t)| = 1$.*

2. *The integral*

$$\int_0^1 S_{i_1}^{[n]}(t) \ldots S_{i_n}^{[n]}(t)\mathrm{d}t = \begin{cases} 1, & \text{if } i_1 = \cdots = i_n \\ 0 & \text{otherwise.} \end{cases}$$

* Corresponding author.
E-mail addresses: icherneha@ukr.net (I. Chernega), andriyzag@yahoo.com (A. Zagorodnyuk).

0040-9383/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.top.2009.11.019

3. *If $j_1, \ldots, j_k$ are distinct positive integers, then for $\sigma_j^{[n]}(t) = S_j^{[n]}(t)$ or $\sigma_j^{[n]}(t) = \overline{S_j^{[n]}(t)}$,*

$$\int_0^1 (\sigma_{j_1}^{[n]})^{m_1}(t) \ldots (\sigma_{j_k}^{[n]})^{m_k}(t)\mathrm{d}t = \begin{cases} 1, & \text{if } m_1 \equiv \cdots \equiv m_k \equiv 0 \ (\mathrm{mod} n) \\ 0 & \text{otherwise.} \end{cases}$$

A mapping $P : X \to Y$ is said to be a *polynomial of degree n* if it can be represented as a sum

$$P(x) = \sum_{k=0}^{n} P_k(x),$$

where P_k is an k-homogeneous polynomial, $1 \le k \le n$, $P_0 \in Y$ and $P_n \not\equiv 0$.

Let now X, Y be complex Banach spaces and Ω be an open subset of X. A mapping $f : \Omega \to Y$ is said to be *analytic* if for any $x_0 \in \Omega$ there exists a neighborhood of x_0, $V_{x_0} \subset \Omega$ such that for every $x \in V_{x_0}$

$$f(x) = \sum_{k=0}^{\infty} f_k(x),$$

where f_k is a k-homogeneous polynomial and the series converges uniformly on V_{x_0}.

It is well known that polynomials f_k are kth Frechet derivatives,

$$f_k = \frac{d^k(x)f}{n!}$$

of f at x_0 by direction x. If $\Omega = X$ we say that f is an *entire mapping*.

Let $f : \Omega \to Y$ be an analytic mapping, $x \in \Omega$ and B be the unit ball of X. The *radius of uniform convergence* $\varrho_x(f)$ of f at x is defined as supremum of λ, $\lambda \in \mathbb{C}$ such that $x + \lambda B \subset \Omega$ and the Taylor series of f at x converges to f uniformly on $x + \lambda B$. The *radius of boundedness* of f at x is defined as supremum of $\lambda \in \mathbb{C}$, such that f is bounded on $x + \lambda B$.

Theorem 1.2 (*[6]*). *The radius of uniform convergence of an entire mapping f at zero coincides with the radius of boundedness of f at zero and*

$$\varrho_0(f) := \left(\limsup_{n\to\infty} \|f_n\|^{1/n}\right)^{-1}. \tag{1}$$

We say that $f : X \to Y$ is an *entire mapping of bounded type* if f is bounded on all bounded subsets (i.e. has the radius of boundedness equal to infinity.)

2. Polarization formula for polynomials

Let X and Y be Banach spaces. Let us denote by $L_a\left({}^nX, Y\right)$ the space of all symmetric n-linear mappings

$$A : \underbrace{X \times \cdots \times X}_{n} \to Y.$$

Let $P_a\left({}^nX, Y\right)$ denote the space of all n-homogeneous polynomials $P : X \to Y$. For any $P \in P_a\left({}^nX, Y\right)$ there exists a unique element $A \in L_a\left({}^nX, Y\right)$ such that $P(x) = A(x, \ldots, x)$. To obtain A from P we can apply the polarization formula with the generalized Rademacher functions (see [5]):

$$A\left(x_1^{n_1}, \ldots, x_k^{n_k}\right) = \frac{n_1! \ldots n_k!}{n!} \int_0^1 \left(S_1^{[n]}\right)^{n-n_1}(t) \ldots \left(S_n^{[n]}\right)^{n-n_k}(t) P\left(S_1^{[n]}(t)x_1 + \cdots + S_k^{[n]}(t)x_k\right) \mathrm{d}t, \tag{2}$$

where $n_1 + \cdots + n_k = n$, $n_1, \ldots, n_k$ are non-negative integers and

$$A\left(x_1^{n_1}, \ldots, x_k^{n_k}\right) := A(\underbrace{x_1, \ldots, x_1}_{n_1}, \ldots, \underbrace{x_k, \ldots, x_k}_{n_k}).$$

Putting $n_1, \ldots, n_k = 1$ and $k = n$, then from (2) we obtain

$$A(x_1, \ldots, x_n) = \frac{1}{n!} \int_0^1 \left(S_1^{[n]}\right)^{n-1}(t) \ldots \left(S_n^{[n]}\right)^{n-1}(t) P\left(S_1^{[n]}(t)x_1 + \cdots + S_n^{[n]}(t)x_n\right) \mathrm{d}t.$$

Let us set

$$\Pi_k(P)(x_1, \ldots, x_k) = \frac{1}{k!} \int_0^1 \left(S_1^{[k]}\right)^{k-1}(t) \ldots \left(S_k^{[k]}\right)^{k-1}(t) P\left(S_1^{[k]}(t)x_1 + \cdots + S_k^{[k]}(t)x_k\right) \mathrm{d}t$$

and let us introduce a set $\mathbb{N}_i \subset \mathbb{N}$ by $\mathbb{N}_i = \{p = p_1 p_2 \ldots p_i : p_1 < p_2 < \cdots < p_i \text{ are prime numbers}\}$.

Theorem 2.1. *Let $P = P_0 + \cdots + P_n$ be a polynomial of degree n and A_k be a symmetric k-linear mapping such that $P_k(x) = A_k(x, \ldots, x)$ for some $1 \le k \le n$. Then*

$$A_k\left(x_1, \ldots x_k\right) = \varPi_k(P)(x_1, \ldots, x_k) + \sum_{i=1}^{r}(-1)^i \sum_{\mathbb{N}_i \ni p \le r} \varPi_{pk}\left(P\right)\left(x_1^p, \ldots, x_k^p\right), \tag{3}$$

where $r = [\frac{n}{k}]$.

Proof. Let us first show that $A_n\left(x_1, \ldots, x_n\right) = \varPi_n(P)(x_1, \ldots, x_n)$. Since the polynomial P is a sum of k-homogeneous polynomials P_k, $1 \le k \le n$, we have

$$\begin{aligned}
&\frac{1}{n!}\int_0^1 \left(S_1^{[n]}\right)^{n-1}(t)\ldots\left(S_n^{[n]}\right)^{n-1}(t)P\left(S_1^{[n]}(t)x_1 + \cdots + S_n^{[n]}(t)x_n\right)\mathrm{d}t\\
&= \frac{1}{n!}\int_0^1 \left(S_1^{[n]}\right)^{n-1}(t)\ldots\left(S_n^{[n]}\right)^{n-1}(t)\sum_{k=0}^{n} P_k\left(S_1^{[n]}(t)x_1 + \cdots + S_n^{[n]}(t)x_n\right)\mathrm{d}t\\
&= \frac{1}{n!}P_0\int_0^1 \left(S_1^{[n]}\right)^{n-1}(t)\ldots\left(S_n^{[n]}\right)^{n-1}(t)\mathrm{d}t + \frac{1}{n!}\int_0^1 \left(S_1^{[n]}\right)^{n-1}(t)\ldots\left(S_n^{[n]}\right)^{n-1}(t)\sum_{m=1}^{n} S_m^{[n]}(t)A_1\left(x_m\right)\mathrm{d}t + \cdots\\
&\quad + \frac{1}{n!}\int_0^1 \left(S_1^{[n]}\right)^{n-1}(t)\ldots\left(S_n^{[n]}\right)^{n-1}(t)\sum_{m_1,\ldots,m_k=1}^{n} S_{m_1}^{[n]}(t)\ldots S_{m_k}^{[n]}(t)A_k\left(x_{m_1}, \ldots, x_{m_k}\right)\mathrm{d}t + \cdots\\
&\quad + \frac{1}{n!}\int_0^1 \left(S_1^{[n]}\right)^{n-1}(t)\ldots\left(S_n^{[n]}\right)^{n-1}(t)\sum_{m_1,\ldots,m_n=1}^{n} S_{m_1}^{[n]}(t)\ldots S_{m_n}^{[n]}(t)A_n\left(x_{m_1}, \ldots, x_{m_n}\right)\mathrm{d}t.
\end{aligned}$$

By the properties of the generalized Rademacher functions (Proposition 1.1) all of the terms for $0 \le k \le n-1$ are equal to zero. So we have only a term which corresponds to the polynomial P_n and

$$\begin{aligned}
&\frac{1}{n!}\int_0^1 \left(S_1^{[n]}\right)^{n-1}(t)\ldots\left(S_n^{[n]}\right)^{n-1}(t)\sum_{m_1,\ldots,m_n=1}^{n} S_{m_1}^{[n]}(t)\ldots S_{m_n}^{[n]}(t)A_n\left(x_{m_1}, \ldots, x_{m_n}\right)\mathrm{d}t\\
&= \frac{1}{n!}\int_0^1 \left(S_1^{[n]}\right)^{n-1}(t)\ldots\left(S_n^{[n]}\right)^{n-1}(t)P_n\left(S_1^{[n]}(t)x_1 + \cdots + S_n^{[n]}(t)x_n\right)\mathrm{d}t = A_n(x_1, \ldots, x_n).
\end{aligned}$$

Let us now find $A_k\left(x_1, \ldots, x_k\right)$, $1 \le k \le n$. If $0 < m \le k-1$, then by the properties of the generalized Rademacher functions

$$\begin{aligned}
\varPi_k\left(P_m\right)\left(x_1, \ldots, x_k\right) &= \frac{1}{k!}\int_0^1 \left(S_1^{[k]}\right)^{k-1}(t)\ldots\left(S_k^{[k]}\right)^{k-1}(t)P_m\left(S_1^{[k]}(t)x_1 + \cdots + S_k^{[k]}(t)x_k\right)\mathrm{d}t\\
&= \frac{1}{k!}\int_0^1 \left(S_1^{[k]}\right)^{k-1}(t)\ldots\left(S_k^{[k]}\right)^{k-1}(t)\\
&\qquad \times \sum_{m_1,\ldots,m_m=1}^{k} S_{m_1}^{[k]}(t)\ldots S_{m_m}^{[k]}(t)A_m\left(x_{m_1}, \ldots, x_{m_m}\right)\mathrm{d}t = 0.
\end{aligned}$$

Using the same properties, we have

$$\begin{aligned}
\varPi_k\left(P\right)\left(x_1, \ldots, x_k\right) &= \frac{1}{k!}\int_0^1 \left(S_1^{[k]}\right)^{k-1}(t)\ldots\left(S_k^{[k]}\right)^{k-1}(t)P\left(S_1^{[k]}(t)x_1 + \ldots + S_k^{[k]}(t)x_k\right)\mathrm{d}t\\
&= \frac{1}{k!}\int_0^1 \left(S_1^{[k]}\right)^{k-1}(t)\ldots\left(S_k^{[k]}\right)^{k-1}(t)P_k\left(S_1^{[k]}(t)x_1 + \cdots + S_k^{[k]}(t)x_k\right)\mathrm{d}t\\
&\quad + \frac{1}{k!}\int_0^1 \left(S_1^{[k]}\right)^{k-1}(t)\ldots\left(S_k^{[k]}\right)^{k-1}(t)P_{2k}\left(S_1^{[k]}(t)x_1 + \ldots + S_k^{[k]}(t)x_k\right)\mathrm{d}t + \cdots\\
&\quad + \frac{1}{k!}\int_0^1 \left(S_1^{[k]}\right)^{k-1}(t)\ldots\left(S_k^{[k]}\right)^{k-1}(t)P_{rk}\left(S_1^{[k]}(t)x_1 + \cdots + S_k^{[k]}(t)x_k\right)\mathrm{d}t,
\end{aligned}$$

where $r = [\frac{n}{k}]$. So we obtain that

$$\varPi_k\left(P\right)\left(x_1, \ldots, x_k\right) = A_k\left(x_1, \ldots, x_k\right) + A_{2k}\left(x_1^2, \ldots, x_k^2\right) + \cdots + A_{rk}\left(x_1^r, \ldots, x_k^r\right), \tag{4}$$

that is,

$$A_k\left(x_1,\ldots,x_k\right)=\Pi_k\left(P\right)\left(x_1,\ldots,x_k\right)-A_{2k}\left(x_1^2,\ldots,x_k^2\right)-\cdots-A_{rk}\left(x_1^r,\ldots,x_k^r\right). \tag{5}$$

Using the same arguments,

$$A_{2k}\left(x_1^2,\ldots,x_k^2\right)=\Pi_{2k}\left(P\right)\left(x_1^2,\ldots,x_k^2\right)-A_{4k}\left(x_1^4,\ldots,x_k^4\right)-\cdots-A_{2mk}\left(x_1^{2m},\ldots,x_k^{2m}\right),$$

where $m=[\frac{n}{2k}]$;

$$A_{3k}\left(x_1^3,\ldots,x_k^3\right)=\Pi_{3k}\left(P\right)\left(x_1^3,\ldots,x_k^3\right)-A_{6k}\left(x_1^6,\ldots,x_k^6\right)-\cdots-A_{3pk}\left(x_1^{3p},\ldots,x_k^{3p}\right),$$

where $p=[\frac{n}{3k}]$ and so on. Substituting all A_{tk}, $2\le t\le r$ in (5) and grouping corresponding terms we have

$$\begin{aligned}A_k\left(x_1,\ldots x_k\right) &= \Pi_k(P)(x_1,\ldots,x_k)-\sum_{p\in\mathbb{N}_1}\Pi_{pk}\left(P\right)\left(x_1^p,\ldots,x_k^p\right)\\ &\quad+\sum_{p\in\mathbb{N}_2}\Pi_{pk}\left(P\right)\left(x_1^p,\ldots,x_k^p\right)+\cdots+\sum_{p\in\mathbb{N}_r}(-1)^r\,\Pi_{pk}\left(P\right)\left(x_1^p,\ldots,x_k^p\right)\\ &= \Pi_k(P)(x_1,\ldots,x_k)+\sum_{i=1}^{r}\sum_{\mathbb{N}_i\ni p\le r}(-1)^i\,\Pi_{pk}\left(P\right)\left(x_1^p,\ldots,x_k^p\right),\end{aligned}$$

where $\mathbb{N}_i=\{p=p_1p_2\ldots p_i : p_1<p_2<\cdots<p_i$ are prime numbers$\}$ and $r=[\frac{n}{k}]$.

Note that $\Pi_{pk}(P)=0$ if $p\equiv 0 \pmod{p_j^m}$, $m>1$, where p_j is an arbitrary prime number. □

Remark 2.2. Eq. (4) may be understood as a system of n linear nonhomogeneous equations with n variables $A_k=A_k(x_1,\ldots,x_k)$, $1\le k\le n$, $x_1,\ldots,x_k$ are fixed. The corresponding homogeneous system can be represented by a matrix C with elements (c_{ij}):

$$c_{ij}=\begin{cases}1, & j\equiv 0 \pmod i;\\ 0, & \text{otherwise.}\end{cases}$$

By the Kronecker–Capelli Theorem there exists a unique solution of the system. Variables A_k, $1\le k\le n$ of system (4) are determined by

$$A_k=\frac{\det C_k}{\det C}, \tag{6}$$

where C_k is a matrix obtained from the matrix C by replacing the kth column by the column $(\Pi_1(P)(x_1),\ldots,\Pi_n(P)(x_1,\ldots,x_n))^\intercal$. Since C is a triangular matrix with units on the diagonal, we can write

$$A_k=\det C_k.$$

Note that $A_k=\Pi_k(P)$ if $k>\frac{n}{2}$.

Corollary 2.3. *Let $P=P_0+\cdots+P_n$ be an arbitrary polynomial of degree n on X, where $P_0\equiv const$ and P_k are k-homogeneous polynomials for $k=1,\ldots,n$. Let A_n be a symmetric n-linear form, which generates P_n. Then*

$$A_n\left(x_1,\ldots,x_n\right)=\frac{1}{n!}\int_0^1\left(S_1^{[n]}\right)^{n-1}(t)\ldots\left(S_n^{[n]}\right)^{n-1}(t)P\left(S_1^{[n]}(t)x_1+\cdots+S_n^{[n]}(t)x_n\right)\mathrm{d}t.$$

Corollary 2.4. *For any $P\in P_a\left({}^nX,Y\right)$*

$$\Pi_k(P)(x_1,\ldots,x_k)=\sum_{m=1}^{\infty}A_{mk}(x_1^m,\ldots,x_k^m).$$

3. The case of analytic mappings

Definition 3.1. We say that a mapping $B : X^k\to Y$ is (k,m)-linear if $B(x_1,\ldots,x_k)$ is an m-homogeneous polynomial with respect to every single variable x_j for another $x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_k$ fixed. The mapping B is called symmetric if $B(x_1,\ldots,x_k)=B(x_{\sigma(1)},\ldots,x_{\sigma(k)})$ for any permutation σ on the set $\{1,\ldots,k\}$.

It is easy to see that for a symmetric (k,m)-linear mapping B there exists an unique symmetric km-linear mapping A such that

$$B(x_1,\dots,x_k)=A(x_1^m,\dots,x_k^m).$$

Let now X,Y be Banach spaces and $L\left({}^nX,Y\right)$, $P\left({}^nX,Y\right)$ be spaces of continuous n-linear mappings from X^n into Y and continuous n-homogeneous polynomials from X into Y respectively. Let us denote by $L({}^k_mX,Y)$ the space of continuous (k,m)-linear mappings. Note that the spaces $L({}^nX,Y)$, $L({}^k_mX,Y)$ and $P({}^nX,Y)$ are Banach spaces with sup-norm over unit balls of corresponding spaces.

Proposition 3.2. *The mapping Π_k is a continuous linear operator from $\mathcal{P}({}^nX,Y)$ into $\mathcal{L}({}^k_mX,Y)$.*

Proof. The operator Π_k is linear by the definition. Let us show the continuity of Π_k. Let $P\in\mathcal{P}({}^nX,Y)$ for some n. If k does not divide n, then $\Pi_k(P)=0$. Suppose that $n=km$ for some m. Since by (4)

$$\Pi_k(P)(x_1,\dots,x_k)=\sum_{s=1}^{m}A_{sk}(x_1^s,\dots,x_k^s)$$
$$m\|P\|\le\|\Pi_k(P)\|\le m\|A\|,$$

where $A(x,\dots,x)=P(x)$. On the other hand, by the polarization inequality [7, p. 10],

$$\|A\|\le\frac{n^n}{n!}\|P\|.$$

So Π_k is a bounded linear operator on the Banach space $\mathcal{P}({}^nX,Y)$. Hence it is continuous. □

Lemma 3.3. *Let $f=\sum_{n=0}^{\infty}f_n$ be an entire mapping of bounded type from X to Y. Then for every k there is a well defined mapping $\Pi_k(f):X^k\to Y$, such that*

$$\begin{aligned}\Pi_k(f)(x_1,\dots,x_k)&:=\lim_{m\to\infty}\sum_{n=0}^{m}\Pi_k(f_n)(x_1,\dots,x_k)\\&=\sum_{n=0}^{\infty}\Pi_k(f_n)(x_1,\dots,x_k)\end{aligned}$$

and the series on the right converges for any $x_1,\dots,x_k\in X$.

Proof. Since f is an entire mapping of bounded type on X, by (1)

$$\limsup_{n\to\infty}\|f_n\|^{1/n}=0.$$

For any fixed $x_1,\dots,x_k\in X$ such that $\max(\|x_1\|,\dots,\|x_k\|)\neq 0$ consider a formal series

$$\sum_{n=0}^{\infty}\Pi_k(f_n)(tz_1,\dots,tz_k)=\sum_{n=0}^{\infty}t^n\Pi_k(f_n)(z_1,\dots,z_k),\tag{7}$$

where $z_k=\frac{x_k}{\max(\|x_1\|,\dots,\|x_k\|)}$ and $t\in\mathbb{C}$.

By the Cauchy–Hadamard formula the radius of convergence of the series at zero is

$$\varrho_0=\left(\limsup_{n\to\infty}\|\Pi_k(f_n)(z_1,\dots,z_k)\|^{1/n}\right)^{-1}.$$

Using the Stirling formula and inequality $\|\Pi_k(f_n)\|\le\frac{n^n}{n!}\|f_n\|$ we have

$$\begin{aligned}\|\Pi_k(f_n)(z_1,\dots,z_k)\|^{1/n}&\le\left(\frac{n^n}{n!}\|f_n\|\max(\|z_1\|,\dots,\|z_k\|)\right)^{1/n}\\&\le e\max\Big(\|z_1\|,\dots,\|z_k\|\Big)^{1/n}\|f_n\|^{1/n}.\end{aligned}$$

So

$$\frac{1}{\varrho_0}=\limsup_{n\to\infty}\|\Pi_k(f_n)(z_1,\dots,z_k)\|^{1/n}=0.$$

Hence the series (7) converges for every t and so in particular it is convergent for $t=\max(\|x_1\|,\dots,\|x_k\|)$ as well. □

Theorem 3.4. *Let A_k be a k-linear symmetric mapping corresponding to k-homogeneous component f_k of entire mapping of bounded type f. Then*

$$A_k(x_1,\dots x_k) = \Pi_k(f)(x_1,\dots,x_k) + \sum_{i=1}^{\infty}(-1)^i \sum_{p\in\mathbb{N}_i} \Pi_{pk}(f)\left(x_1^p,\dots,x_k^p\right), \tag{8}$$

where $\mathbb{N}_i = \{p = p_1p_2\dots p_i : p_1 < p_2 < \cdots < p_i$ *are prime numbers*$\}$.

Proof. For the simplicity we rewrite formula (8) by

$$A_k(x_1,\dots x_k) = \Pi_k(f)(x_1,\dots,x_k) + \sum_{i=2}^{\infty} c_i \Pi_{ik}(f)\left(x_1^i,\dots,x_k^i\right),$$

where $c_i = 0;\ 1;\ -1$.

By Theorem 2.1 for an arbitrary n

$$A_k(x_1,\dots x_k) = \Pi_k\left(\sum_{j=0}^{n} f_j\right)(x_1,\dots,x_k) + \sum_{i=2}^{r} c_i\Pi_{ik}\left(\sum_{j=0}^{n} f_j\right)\left(x_1^i,\dots,x_k^i\right),$$

where $r = [\frac{n}{k}]$. Using Proposition 3.2 we obtain that

$$A_k(x_1,\dots x_k) = \sum_{j=0}^{n}\Pi_k(f_j)(x_1,\dots,x_k) + \sum_{i=2}^{r}\sum_{j=0}^{n} c_i\Pi_{ik}(f_j)\left(x_1^i,\dots,x_k^i\right).$$

Proceeding to the limit as $n\to\infty$ and observing that

$$\sum_{i=2}^{r}\sum_{j=0}^{n} c_i\Pi_{ik}(f_j) = \sum_{i=2}^{\infty}\sum_{j=0}^{n} c_i\Pi_{ik}(f_j),$$

we obtain

$$A_k(x_1,\dots,x_k) = \lim_{n\to\infty}\sum_{j=0}^{n}\Pi_k(f_j)(x_1,\dots,x_k) + \sum_{i=2}^{\infty}\lim_{n\to\infty}\sum_{j=0}^{n} c_i\Pi_{ik}(f_j)\left(x_1^i,\dots,x_k^i\right).$$

By Lemma 3.3,

$$\begin{aligned} A_k(x_1,\dots,x_k) &= \Pi_k\left(\sum_{j=0}^{\infty} f_j\right)(x_1,\dots,x_k) + \sum_{i=2}^{\infty} c_i\Pi_{ik}\left(\sum_{j=0}^{\infty} f_j\right)\left(x_1^i,\dots,x_k^i\right) \\ &= \Pi_k(f)(x_1,\dots,x_k) + \sum_{i=2}^{\infty} c_i\Pi_{ik}(f)\left(x_1^i,\dots,x_k^i\right). \quad \square \end{aligned}$$

References

[1] H.F. Bohnenblust, E. Hille, On the absolute convergence of Dirichlet series, Ann. of Math. 32 (2) (1931) 610.
[2] R.S. Martin, Contributions to the theory of functionals, Ph.D. thesis, University of California, 1932. Unpublished.
[3] S. Mazur, W. Orlicz, Grundlegende eigenschaften der polynomischen operationen I, II, Studia Math. 5 (1935) 50–68. 179–189.
[4] R.M. Aron, J. Globevnik, Analytic functions on c_0, Rev. Mat. 2 (1989) 27–34.
[5] R.M. Aron, M. Lacruz, R.A. Ryan, A.M. Tonge, The generalized Rademacher Functions, Note Math. 12 (1992) 15–22.
[6] S. Dineen, Complex Analysis in Locally Convex Spaces, in: Mathematics Studies, vol. 57, North-Holland, Amsterdam, New York, Oxford, 1981.
[7] S. Dineen, Complex Analysis on Infinite Dimensional Spaces, in: Monographs in Mathematics, Springer, New York, 1999.